Global solutions below the energy class (Variational Problems and Related Topics)

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1 Introduction and Main results

In this note, we consider the system of Klein-Gordon-Schrödinger equations with Yukawa coupling:

$$\begin{cases} i\partial_t u+\Delta u &= 2vu, \quad x\in\mathbb{R}^d, \quad t\geq 0, \\ \partial_t^2 v-\Delta v+v &= -|u|^2, \quad x\in\mathbb{R}^d, \quad t\geq 0, \end{cases} \tag{1}$$

which represents the classical model of dynamics of conserved complex nucleon field u interacting with neutral real scalar meson field v.

We are interested in the global well-posedness of the Cauchy problem for this system, especially, when data do not have the finite energy.

Global well-posedness below the energy class is recently developed by J. Bourgain [3, 4] and J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao [5, 6, 7]. In [12], H. Pecher has proved that, if $d=3$ and $1\geq s_1, s_2>7/10$ with $s_1+s_2>3/2$, then the system (1) is globally well-posed for the data $(u(0), v(0), v_t(0))\in H^{s_1}\times H^{s_2}\times H^{s_2-1}$. His proof is based on the idea of Bourgain.

Our aim here is to extend his result, in particular, to the high dimensional case $d=4$. We obtain the following result: Let $d\leq 4$. Assume (4) for $u(0)$ when $d=4$. If $1\geq s_1, s_2>4/(8+2s_2-d)$, then (1) is globally well-posed for the data $(u(0), v(0), v_t(0))\in H^{s_1}\times H^{s_2}\times H^{s_2-1}$. Our proof is based on the I-method [5]. But we encounter the complicated high-low frequency interactions caused by the system, which do not appear in single equations such as the KdV and the Schrödinger equations [5, 6, 7]. To analyze these interactions, we use the conservation of the energy represented by the Bourgain weight (see the case (2-3) in the section 4).

Moreover, introducing the space wihch controls the low frequency part and the modified multiplier for I-method, we obtain the similar result for the massless version of (1) which is the wave-Schrödinger system below (see Theorem 6.1).

$$\begin{cases} i\partial_t u+\Delta u &= 2vu, \\ \partial_t^2 v-\Delta v &= -|u|^2, \end{cases}$$

where u and v are complex and real valued functions on $\mathbb{R}^d\times[0,\infty)$, respectively.

The Klein-Gordon-Schrödinger system above is transformed into a time first oreder system in the usual way [8, 12] and so, in what follows, we consider the following Cauchy problem.

$$
\text{(KGS)}\begin{cases} i\partial_t\psi + \Delta\psi &= (\phi+\overline{\phi})\psi, & x\in\mathbb{R}^d, \quad t\geq 0,\\ i\partial_t\phi - (1-\Delta)^{\frac{1}{2}}\phi &= (1-\Delta)^{-\frac{1}{2}}(|\psi|^2), & x\in\mathbb{R}^d, \quad t\geq 0,\\ \psi(0) &= \psi_0\in H^{s_1}(\mathbb{R}^d), & x\in\mathbb{R}^d,\\ \phi(0) &= \phi_0\in H^{s_2}(\mathbb{R}^d), & x\in\mathbb{R}^d,\end{cases}
$$

where both ψ and ϕ are complex valued functions.

For (KGS), we formally have the mass and the Hamiltonian conservation laws:

$$
\|\psi(t)\|_{L^2(\mathrm{R}^d)} = \|\psi_0\|_{L^2(\mathrm{R}^d)}, \tag{2}
$$

$$
H(\psi(t),\phi(t)) = H(\psi_0,\phi_0), \tag{3}
$$

where

$$
H(f,g) := \|f\|^2_{\dot{H}^1(\mathrm{R}^d)} + \|g\|^2_{H^1(\mathrm{R}^d)} + \int_{\mathrm{R}^d}(g(x)+\overline{g}(x))|f(x)|^2\,dx.
$$

From (2) and (3), it follows that (KGS) is globally well-posed if $d\leq 3$ and $s_1=s_2=1$. Moreover, if $d=4$, $s_1=s_2=1$ and

$$
\|\psi_0\|_{L^2(\mathrm{R}^d)} < \frac{(S_4)^{\frac{1}{2}}}{(G_{\frac{1}{3},4})^{\frac{3}{4}}}, \tag{4}
$$

then (KGS) is globally well-posed, where S_d and $G_{\sigma,d}$ are respectively the best constants of the Sobolev and the Gagliardo-Nirenberg inequalities:

$$
S_d\|f\|_{L^{\frac{2d}{d-2}}(\mathrm{R}^d)} \leq \|\nabla f\|^2_{L^2(\mathrm{R}^d)},
$$

$$
\|f\|^{2\sigma+2}_{L^{2\sigma+2}(\mathrm{R}^d)} \leq G_{\sigma,d}\|\nabla f\|^{\sigma d}_{L^2(\mathrm{R}^d)}\|f\|^{2\sigma+2-\sigma d}_{L^2(\mathrm{R}^d)}, \quad 0<\sigma<\frac{2}{d-2} \quad (d\geq 2).
$$

Our main result is as follows.

Theorem 1.1 (Global well-posedness)
Let $d\leq 4$, and assume (4) when $d=4$. If s_1 and s_2 satisfy that

$$
1\geq s_1, s_2 > \frac{4}{8+2s_2-d}, \tag{5}
$$

then (KGS) is globally well-posed.

Remark 1

(i) From the Lemma 1.2 below, we find that (KGS) is locally well-posed under the conditions of Theorem 1.1.

(ii) As stated above, in [12], H. Pecher has proved the following: If $d = 3$ and $1 \geq s_1, s_2 > 7/10$ with $s_1 + s_2 > 3/2$, then (KGS) is globally well-posed. Our result is an extension of [12]. We briefly refer to the Pecher approach in the section 2 as the known results.

To prove Theorem 1.1, the Bourgain spaces are essential and therefore we first introduce them. After that, we give Lemma 1.2, which will play a crucial role for the proof of Theorem 1.1.

Let U and V denote the free evolution operators of Schrödinger and Klein-Gordon equations, respectively, i.e. $U = e^{it\Delta} := \mathcal{F}_\xi^{-1} e^{-it|\xi|^2} \mathcal{F}_x$ and $V = e^{it(1-\Delta)^{\frac{1}{2}}} := \mathcal{F}_\xi^{-1} e^{it\langle\xi\rangle} \mathcal{F}_x$ where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ and $\mathcal{F}_z, \mathcal{F}_z^{-1}$ denote the Fourier and the inverse Fourier transforms with respect to z, respectively.

We define the Bourgain norms and the Bourgain spaces for Schrödinger equations by

$$\|u\|_{X^{s,\alpha}} := \|(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}} U(-\cdot)u\|_{L^2_{x,t}},$$

and

$$X^{s,\alpha} := \left\{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) \mid \|u\|_{X^{s,\alpha}} < \infty \right\}$$

were $\mathcal{S}'$ denotes the class of the tempered distributions. Let L be an interval in $\mathbb{R}$. We define the time-localized space of $X^{s,\alpha}$ by

$$X^{s,\alpha}(L) := \left\{ u : \mathbb{R}^d \times L \to \mathbb{C} : \text{measurable} \,\middle|\, {}^\exists \widetilde{u} \in X^{s,\alpha} \text{ s.t. } \widetilde{u}|_L = u \right\},$$

and its norm by

$$\|u\|_{X^{s,\alpha}(L)} := \inf_{\substack{u \in X^{s,\alpha} \\ u|_L = u}} \|\widetilde{u}\|_{X^{s,\alpha}}.$$

Similarly we introduce the spaces for the Klein-Gordon equation.

$$\|v\|_{Y^{s,\alpha}} = \|(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}} V(-\cdot)v\|_{L^2_{x,t}},$$

$$Y^{s,\alpha} := \left\{ v \in \mathcal{S}'(\mathbb{R}^{d+1}) \mid \|v\|_{Y^{s,\alpha}} < \infty \right\},$$

$$Y^{s,\alpha}(L) := \left\{ v : \mathbb{R}^d \times L \to \mathbb{C} : \text{measurable} \,\middle|\, {}^\exists \widetilde{v} \in Y^{s,\alpha} \text{ s.t. } \widetilde{v}|_L = v \right\},$$

$$\|v\|_{Y^{s,\alpha}(L)} := \inf_{\substack{v \in Y^{s,\alpha} \\ v|_L = v}} \|\widetilde{v}\|_{Y^{s,\alpha}}.$$

By direct calculation, we find that

$$(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}}U(-\cdot)u = U(-\cdot)\mathcal{F}_{\xi,\tau}^{-1}[\langle\xi\rangle^s\langle\tau+|\xi|^2\rangle^\alpha\mathcal{F}_{x,t}[u]] \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^{d+1}) \tag{6}$$

and

$$(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}}V(-\cdot)v = V(\cdot)\mathcal{F}_{\xi,\tau}^{-1}[\langle\xi\rangle^s\langle\tau+\langle\xi\rangle\rangle^\alpha\mathcal{F}_{x,t}[v]] \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^{d+1}). \tag{7}$$

From (6) and (7), it follows that

$$\|u\|_{X^{s,\alpha}} = \|\langle\xi\rangle^s\langle\tau+|\xi|^2\rangle^\alpha\mathcal{F}_{x,t}[u]\|_{L^2_{\xi,\tau}} \tag{8}$$

and

$$\|v\|_{Y^{s,\alpha}} = \|\langle\xi\rangle^s\langle\tau+\langle\xi\rangle\rangle^\alpha\mathcal{F}_{x,t}[v]\|_{L^2_{\xi,\tau}}. \tag{9}$$

Now we set

$$\|u\|_{X_-^{s,\alpha}} := \|(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}}Uu\|_{L^2_{x,t}} = \|\langle\xi\rangle^s\langle\tau-|\xi|^2\rangle^\alpha\mathcal{F}_{x,t}[u]\|_{L^2_{\xi,\tau}},$$

$$\|v\|_{Y_-^{s,\alpha}} := \|(1-\Delta)^{\frac{s}{2}}(1-\partial_t^2)^{\frac{\alpha}{2}}Vv\|_{L^2_{x,t}} = \|\langle\xi\rangle^s\langle\tau-\langle\xi\rangle\rangle^\alpha\mathcal{F}_{x,t}[v]\|_{L^2_{\xi,\tau}}$$

and

$$X_-^{s,\alpha} := \left\{u\in\mathcal{S}'(\mathbb{R}^{d+1}) \,\middle|\, \|u\|_{X_-^{s,\alpha}} < \infty\right\},$$

$$Y_-^{s,\alpha} := \left\{v\in\mathcal{S}'(\mathbb{R}^{d+1}) \,\middle|\, \|v\|_{Y_-^{s,\alpha}} < \infty\right\}.$$

Then we easily see that if $\psi \in X^{s,\alpha}$, then $\overline{\psi} \in X_-^{s,\alpha}$ with the identity

$$\|\overline{\psi}\|_{X_-^{s,\alpha}} = \|\psi\|_{X^{s,\alpha}}. \tag{10}$$

Also $\|\overline{\phi}\|_{Y_-^{s,\alpha}} = \|\phi\|_{Y^{s,\alpha}}$. Further time-localized versions of $X_-^{s,\alpha}$ and $Y_-^{s,\alpha}$ are defined by the same manner as above.

To state Lemma 1.2, we introduce the time smooth cut-off funciton: Let $\rho \in C^\infty(\mathbb{R};[0,1])$ be such that

$$\rho(t) = \begin{cases} 1 & \text{if} \quad |t| \le 1 \\ 0 & \text{if} \quad |t| \ge 2 \end{cases}.$$

Then we define the time smooth cut-off function by $\rho_T(t) := \rho(t/T)$.

Lemma 1.2 (Bilinear estimates with explicit time power)
Let $0 < T \le 1$. Assume that s_1 and s_2 satisfy that

$$1 \ge s_1 \ge 0, \quad 1 \ge s_2 > \max\left\{0,\ 1-\frac{d}{2}\right\}, \quad s_1 - \frac{s_2}{2} > \frac{d}{4} - \frac{3}{2}, \quad s_2 > \frac{d}{2} - 2,$$

and θ, $\widetilde{\theta}$ satisfy that

$$\theta < \min\left\{1 + \frac{s_2}{2} - \frac{d}{4},\ 1 + \frac{s_2}{2} - \frac{s_1}{2},\ 1\right\}, \quad \widetilde{\theta} < \min\left\{\frac{3}{2} + s_1 - \frac{s_2}{2} - \frac{d}{4},\ 1\right\}.$$

Then there exixt $\alpha, \beta > 1/2$ such that

$$\|(\rho_T u)(\rho_T v)\|_{X^{s_1,\alpha-1}} \le C T^{\theta} \|u\|_{X^{s_1,\alpha}} \|v\|_{Y^{s_2,\beta}}, \tag{11}$$

$$\|(1-\Delta)^{-\frac{1}{2}}[(\rho_T u)(\overline{\rho_T u})]\|_{Y^{s_2,\beta-1}} \le \widetilde{C} T^{\widetilde{\theta}} \|u\|^2_{X^{s_1,\alpha}} \tag{12}$$

where both C and $\widetilde{C}$ are independent of T. In the R.H.S. of (11), we may replace $Y^{s_2,\beta}$ with $Y_-^{s_2,\beta}$.

The proof of Lemma 1.2 is similar to [8].

This note is organized as follows. In section 2, we introduce the known results. In particular, we show the key bilinear estimate for the Pecher approach. In section 3, we introduce the smoothing operators and the modified energy of (KGS). Here we give the increment of the modified energy, which is stated in Proposition 3.2. In section 4, we prove the Proposition 3.2. In section 5, we prove the Thorem 1.1. Finally, in section 6, we consider the massless case, the wave-Schrödinger system, briefly.

2 Known results

As stated above, H. Pecher proved the following theorem using Bourgain's idea [3, 4].

Theorem 2.1
Let $d = 3$ and $1 \ge s_1, s_2 > 7/10$ with $s_1 + s_2 > 3/2$. Then (KGS) is globally well-posed.

In this section, we only show the key bilinear estimate to prove Theorem 2.1. For the proof of the theorem, see the original paper [12].

The key estimate is the following.

Lemma 2.2
Let $M_1 \ge 2$, $M_2 > 0$. Suppose that

$$\mathit{supp}\, \mathcal{F}_x[f] \subset \left\{\frac{M_1}{2} \le |\xi| \le 2M_1\right\}, \quad \mathit{supp}\, \mathcal{F}_x[g] \subset \left\{\frac{M_2}{2} \le |\xi| \le 2M_2\right\}.$$

Then

$$\|(Uf)(Vg)\|_{L^2_t L^2_x} \le C \frac{M_2^{\frac{d-1}{2}}}{M_1^{\frac{1}{2}}} \|f\|_{L^2_x} \|g\|_{L^2_x},$$

where C is a constant depending only on the space dimension d.

To porve the Lemma 2.2, we need the following.

Lemma 2.3 (Co-area formula)
Suppose that $P \in C^\infty(\mathbb{R}^d;\mathbb{R})$ *and* $f \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$ *with* $\nabla P \neq 0$ *on* $\operatorname{supp} f$. *Then*

$$\int_{\mathbb{R}^d} f(x)\,\delta(P(x))dx = \int_{\{P(x)=0\}} f(x)\,\frac{d\sigma}{|\nabla P(x)|}.$$

Now we give the proof of Lemma 2.2
Proof of Lemma 2.2.
In what follows, we denote all constants depending only on the space dimension d by C.
First note that

$$(Uf)(t)(Vg)(t) = \mathcal{F}_\xi^{-1}\left[e^{-it|\xi|^2}\mathcal{F}_x[f] * e^{it\langle\xi\rangle}\mathcal{F}_x[g]\right].$$

Then, by Plancherel's theorem with respect to space-time, we have

$$\begin{aligned}
\|(Uf)(Vg)\|_{L_t^2L_x^2} &= \left\|\mathcal{F}_t\left[e^{-it|\xi|^2}\mathcal{F}_x[f] * e^{it\langle\xi\rangle}\mathcal{F}_x[g]\right]\right\|_{L_\tau^2L_\xi^2} \\
&= \left\|\int_{\mathbb{R}^d_{\xi_1}} \mathcal{F}_x[f](\xi_1)\mathcal{F}_x[g](\xi-\xi_1)\mathcal{F}_t\left[e^{-it(|\xi|^2-\langle\xi-\xi_1\rangle)}\right]\,d\xi_1\right\|_{L_\tau^2L_\xi^2} \\
&= \left\|\int_{\mathbb{R}^d_{\xi_1}} \mathcal{F}_x[f](\xi_1)\mathcal{F}_x[g](\xi-\xi_1)\delta(\tau+|\xi_1|^2-\langle\xi-\xi_1\rangle)\,d\xi_1\right\|_{L_\tau^2L_\xi^2} \qquad (13)
\end{aligned}$$

Moreover, by Lemma 2.3 and Schwartz inequality with respect to $d\sigma$,

$$\begin{aligned}
\text{R.H.S. of (13)} &= \left\|\int_B \mathcal{F}_x[f](\xi_1)\mathcal{F}_x[g](\xi-\xi_1)\frac{d\sigma}{|\nabla P(\xi_1)|}\right\|_{L_\tau^2L_\xi^2} \\
&\leq |B|^{\frac{1}{2}}\left\|\left(\int_B |\mathcal{F}_x[f](\xi_1)|^2|\mathcal{F}_x[g](\xi-\xi_1)|^2\frac{1}{|\nabla P(\xi_1)|^2}d\sigma\right)^{\frac{1}{2}}\right\|_{L_\tau^2L_\xi^2} \qquad (14)
\end{aligned}$$

where

$$P(\xi_1) = P_{\xi,\tau}(\xi_1) := \tau + |\xi_1|^2 - \langle\xi-\xi_1\rangle,$$

and

$$B = B_{\xi,\tau} := \{P(\xi_1)=0\} \cap \left\{\frac{M_1}{2} \leq |\xi_1| \leq 2M_1\right\} \cap \left\{\frac{M_2}{2} \leq |\xi-\xi_1| \leq 2M_2\right\}.$$

Here we have, for any $\xi \in B$,

$$\frac{M_1}{2} \leq 2|\xi_1| - 1 \leq \left|2\xi_1 - \frac{\xi-\xi_1}{\langle\xi-\xi_1\rangle}\right| = |\nabla P(\xi_1)| \leq 5M_1$$

and thus

$$\begin{aligned}
\text{R.H.S. of (14)} &\leq \frac{\sqrt{2}}{M_1^{\frac{1}{2}}}|B|^{1/2}\left\|\left(\int_B |\mathcal{F}_x[f](\xi_1)|^2|\mathcal{F}_x[g](\xi-\xi_1)|^2\frac{d\sigma}{|\nabla P(\xi_1)|}\right)^{\frac{1}{2}}\right\|_{L^2_\tau L^2_\xi} \\
&\leq C\frac{M_2^{\frac{d-1}{2}}}{M_1^{\frac{1}{2}}}\left\|\left(\int_B |\mathcal{F}_x[f](\xi_1)|^2|\mathcal{F}_x[g](\xi-\xi_1)|^2\frac{d\sigma}{|\nabla P(\xi_1)|}\right)^{\frac{1}{2}}\right\|_{L^2_\tau L^2_\xi} \qquad (15)
\end{aligned}$$

By Lemma 2.3, R.H.S. of (15) is equal to

$$\begin{aligned}
&C\frac{M_2^{\frac{d-1}{2}}}{M_1^{\frac{1}{2}}}\left\|\left(\int_{\mathbb{R}^d_{\xi_1}} |\mathcal{F}_x[f](\xi_1)|^2|\mathcal{F}_x[g](\xi-\xi_1)|^2\delta(\tau+|\xi_1|^2-\langle\xi-\xi_1\rangle)\,d\xi_1\right)^{\frac{1}{2}}\right\|_{L^2_\tau L^2_\xi} \\
= \ &C\frac{M_2^{\frac{d-1}{2}}}{M_1^{\frac{1}{2}}}\|f\|_{L^2_x}\|g\|_{L^2_x},
\end{aligned}$$

which completes the proof. □

At the end of this section, we remark that it seems difficult to apply the key estimate Lemma 2.2 in the high dimensional case $d \geq 4$. Indeed, M_1 and M_2 repesent the freuency supports and therefore differential. In Lemma 2.2, if $d \geq 4$, then the difference of order of M_1 and M_2 is greater than 1, that spoils the same approach as H. Pecher [12].

Thus we employ the I-method without the Lemma 2.2, where I-method is essentialy same as the Bourgain's idea [3, 4].

3 Smoothing operator and Modified energy

In this section, we introduce the operator for the I-method and define the modified energy which makes sense for the functions below the enegy class.

Let $m_N^s \in C^\infty(\mathbb{R}^d;[0,1])$ be radially symmetric, non-increasing and

$$m_N^s(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & \text{if } |\xi| \geq 2N \end{cases} \qquad (16)$$

We set $I_N^s := \mathcal{F}_\xi^{-1} m_N^s \mathcal{F}_x$ and $I_N^1 := 1$.

The properties of I_N^s are stated in the following proposition.

Proposition 3.1 (Properties of I_N^s)
Let $0 \leq s \leq 1$, $2 \leq N$, $s' \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Then we have

$$\|I_N^s f\|_{H^{s'}(\mathbb{R}^d)} \leq \|f\|_{H^{s'}(\mathbb{R}^d)}, \qquad (17)$$

$$\|I_N^s f\|_{\dot{H}^1(\mathbb{R}^d)} \leq 2N^{1-s}\|f\|_{\dot{H}^s(\mathbb{R}^d)}, \qquad (18)$$

$$\|f\|_{H^s(\mathbb{R}^d)} \leq \|I_N^s f\|_{H^1(\mathbb{R}^d)}, \qquad (19)$$

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|I_N^s f\|_{\dot{H}^1(\mathbb{R}^d)}^s\|I_N^s f\|_{L^2(\mathbb{R}^d)}^{1-s} + \|I_N^s f\|_{\dot{H}^1(\mathbb{R}^d)}. \qquad (20)$$

Remark 2
(i) By (18), we find that I_N^s is a smoothing operator of order $1-s$.

In what follows, we assume that $s_1, s_2 \le 1$.

We simply write $I_1 := I_N^{s_1}$ and its Fourier multiplier $m_1 := m_N^{s_1}$. Also $I_2 := I_N^{s_2}$ and $m_2 := m_N^{s_2}$.

We define the modified energy of the Caucy problem (KGS) by

$$E_{1,2}(f,g) := H(I_1 f, I_2 g). \tag{21}$$

For the space-time functions $u = u(x,t), v = v(x,t)$, we simply write

$$E(u,v)(t) := E(u(t), v(t)).$$

If $f \in H^{s_1}$ and $g \in H^{s_2}$, then, by Proposition 3.1, we find that this modified energy is finite, although the Hamiltonian H is not finite for $s_1, s_2 < 1$.

The increment of the modified energy is estimated as follows.

Proposition 3.2
Let $d \le 4$, $N \ge 32$, $L := [t_0, t_1]$, $\alpha, \beta > 1/2$, $\varepsilon > 0$ and (ψ, ϕ) be a $H^{s_1} \times H^{s_2}$-solution of (KGS) on L. Assume that $1 \ge s_1 > 1/2$, $1 \ge s_2 > 0$ with $s_1 + s_2 > 1$. Then we have

$$\begin{aligned}
&E_{1,2}(\psi,\phi)(t_1) - E_{1,2}(\psi,\phi)(t_0)\\
&\le C_* \left\{ \frac{1}{N^{1-\varepsilon}} \left(\|I_N^{s_1}\psi\|_{X^{1,\alpha}(L)} + \|I_N^{s_2}\phi\|_{Y^{1,\beta}(L)} \right)^3 + \frac{1}{N^{\frac{3}{2}-\varepsilon}} \left(\|I_N^{s_1}\psi\|_{X^{1,\alpha}(L)} + \|I_N^{s_2}\phi\|_{Y^{1,\beta}(L)} \right)^4 \right\}
\end{aligned}$$

whrere C_* is independent of $L = [t_0, t_1]$ and N.

The proof of Proposition 3.2 is given in the next section.

4 Proof of Proposition 3.2

In this section, we prove Proposition 3.2.

First note that, for any functions $u \in C(L; H^2(\mathbb{R}^d)) \cap C^1(L; L^2(\mathbb{R}^d))$, $v \in C(L; H^1(\mathbb{R}^d)) \cap C^1(L; L^2(\mathbb{R}^d))$, we have

$$\begin{aligned}
\partial_t H(u(t), v(t)) = &- 2\Re \int_{\mathrm{R}^d} \overline{\partial_t u(x,t)} Eq^{(S)}(u,v)(x,t)\, dt\\
&- 2\Re \int_{\mathrm{R}^d} \overline{(1-\Delta)^{\frac{1}{2}} \partial_t v(x,t)} Eq^{(KG)}(u,v)(x,t)\, dx, \quad \forall t \in L
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
Eq^{(S)}(u,v) &:= i\partial_t u + \Delta u - (v + \overline{v})u,\\
Eq^{(KG)}(u,v) &:= i\partial_t v - (1-\Delta)^{\frac{1}{2}} - (1-\Delta)^{-\frac{1}{2}}(|u|^2).
\end{aligned}$$

Now let (ψ, ϕ) be a solution of (KGS) on $L := [t_0, t_1]$. By the continuous dependence and standard approximation argument, it is enough to prove Proposition 3.2 for the solutions (ψ,ϕ) with $\psi \in C(L;H^2(\mathbb{R}^d)) \cap C^1(L;L^2(\mathbb{R}^d))$ and $\phi \in C(L;H^1(\mathbb{R}^d)) \cap C^1(L;L^2(\mathbb{R}^d))$.

Then, since $E_{1,2}(\psi,\phi)(t) = H(I_1\psi(t), I_2\phi(t))$, by (22) and using the equations,

$$
\begin{aligned}
& E_{1,2}(\psi,\phi)(t_1) - E_{1,2}(\psi,\phi)(t_0) \\
= & \int_L \partial_t E_{1,2}(\psi,\phi)(t)\, dt \\
= & 2\Im \int_L \int_{\mathbb{R}^d} \overline{(-\Delta)I_1\psi}\, \{I_1[(\phi+\overline{\phi})\psi] - (I_2\phi + \overline{I_2\phi})I_1\psi\} && (23)\\
& +2\Im \int_L \int_{\mathbb{R}^d} \overline{I_1[(\phi+\overline{\phi})\psi]}\, \{I_1[(\phi+\overline{\phi})\psi] - (I_2\phi + \overline{I_2\phi})I_1\psi\} && (24)\\
& +2\Im \int_L \int_{\mathbb{R}^d} \overline{(1-\Delta)^{\frac{1}{2}} I_2\phi}\, \{I_2(|\psi|^2) - |I_1\psi|^2\} && (25)\\
& +2\Im \int_L \int_{\mathbb{R}^d} \overline{(1-\Delta)^{-\frac{1}{2}} I_2(|\psi|^2)}\, \{I_2(|\psi|^2) - |I_1\psi|^2\}. && (26)
\end{aligned}
$$

Here, the integrals (23) and (25) are cubic and therefore we want to bound them by

$$\frac{1}{N^{1-\varepsilon}} (\|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)})^3. \tag{27}$$

On the other hand, since the integrals (24) and (26) are quartic, we want to bound them by

$$\frac{1}{N^{\frac{3}{2}-\varepsilon}} (\|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)})^4. \tag{28}$$

The order of differential in (25) and (26) are respectively less than (23) and (24) by 1. Therefore they are easier and we only consider (23) and (24). Moreover, to stress our devise, we concentrate on the estimate of (24).

Thus we consider the integral (24) here. Since the our aim is to show the same bound for all dimension $d \le 4$, we may only consider the case $d = 4$. The other cases are easier. In particular, In the 1 dimensional case, by good bilinear estimate Lemma 2.2, we probably obtain the better order of N and thus Theorem1.1 will be improved.

We denote a smooth dyadic resolution of unity in $\mathbb{R}^d$ by $\{\eta_k\}_{k=0}^{\infty}$, which has the following properties: $\eta_k \in C^\infty(\mathbb{R}^d;[0,1])$ $(k \in \mathbb{N}\cup\{0\})$, $\operatorname{supp}\eta_0 \subset \{|\xi| \le 2\}$, $\operatorname{supp}\eta_k \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}$ $(k \in \mathbb{N})$ and

$$\sum_{k=0}^{\infty} \eta_k(\xi) = 1, \ \forall \xi \in \mathbb{R}^d.$$

Now let us consider (24) in the 4 dimensional case. By Plancherel's theorem in space,

$$
\begin{aligned}
(24) \;=\; & 2\Im \int_L \int_{\xi=\xi_{12}=\xi_{34}} \left(\frac{m_1(\xi)}{m_1(\xi_1)m_2(\xi_2)} - 1 \right) \frac{m_1(\xi)}{m_1(\xi_3)m_2(\xi_4)} \\
& \times \mathcal{F}_x[I_1\psi](\xi_1,t)\mathcal{F}_x[I_2\phi+\overline{I_2\phi}](\xi_2,t)\overline{\mathcal{F}_x[I_1\psi]}(\xi_3,t)\overline{\mathcal{F}_x[I_2\phi+\overline{I_2\phi}]}(\xi_4,t) && (29)
\end{aligned}
$$

where $\int_{\xi=\xi_{12}=\xi_{34}}$ denotes $\int_{\substack{\mathbb{R}^d_\xi\times\mathbb{R}^d_{\xi_1}\times\mathbb{R}^d_{\xi_3} \\ \xi=\xi_1+\xi_2=\xi_3+\xi_4}} d\xi_3 d\xi_1 d\xi$.

In the usual I-method, we start the analysis from now. But, to overcome the difficulty appearing later, we further take the time Fourier transform. So, take arbitrary extensions $\psi_1 \in X^{1,\alpha}$, $\phi_2 \in Y^{1,\beta}$ such that $\psi_1|_L = I_1\psi$, $\phi_2|_L = I_2\phi$ and replace them in (29). Moreover insert the characteristic function χ_L (observe that we can not use the time smooth cut-off function) and use Plancherel's theorem in time . Then we have

$$
\begin{aligned}
&\text{R.H.S. of (29)} \\
&= 2\Im \int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \left(\frac{m_1(\xi)}{m_1(\xi_1)m_2(\xi_2)} - 1\right) \frac{m_1(\xi)}{m_1(\xi_3)m_2(\xi_4)} \\
&\quad \times \mathcal{F}_{x,t}[\psi_1](\xi_1,\tau_1)\mathcal{F}_{x,t}[\phi_2+\overline{\phi_2}](\xi_2,\tau_2)\overline{\mathcal{F}_{x,t}[\psi_1](\xi_3,\tau_3)\mathcal{F}_{x,t}[\chi_L\phi_2+\overline{\chi_L\phi_2}](\xi_4,\tau_4)} \\
&\le 2 \sum_{k_1,k_2,k_3,k_4=0}^{\infty} \int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \prod_{j=1}^{4} \eta_{k_j}(\xi_j) \left\{ M_1M_2 \left|\mathcal{F}_{x,t}[\psi_1](\xi_1,\tau_1)\right| \left|\mathcal{F}_{x,t}[\phi_2+\overline{\phi_2}](\xi_2,\tau_2)\right| \right. \\
&\quad \left. \times \left|\mathcal{F}_{x,t}[\psi_1](\xi_3,\tau_3)\right| \left|\mathcal{F}_{x,t}[\chi_L\phi_2+\overline{\chi_L\phi_2}](\xi_4,\tau_4)\right| \right\} \qquad (30)
\end{aligned}
$$

where $\int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} := \int_{\xi=\xi_{12}=\xi_{34}} \int_{\tau=\tau_{12}=\tau_{34}}$ and $\int_{\tau=\tau_{12}=\tau_{34}}$ is defined as same manner above. Further we put

$$M_1 = M_1(\xi,\xi_1,\xi_2) := \left|\frac{m_1(\xi)}{m_1(\xi_1)m_2(\xi_2)} - 1\right|,$$

$$M_2 = M_2(\xi,\xi_3,\xi_4) := \frac{m_1(\xi)}{m_1(\xi_3)m_2(\xi_4)},$$

and $\{\eta_{k_j}\}_{k_j=0}^{\infty}$ is the dyadic resolution of unity in $\mathbb{R}^d_{\xi_j}$.

We split the different frequency interactions into four cases, according to the size of the parameter N in comparison to the 2^{k_j}:

$$\sum_{k_1,k_2,k_3,k_4} = \sum_{(2-1)} + \sum_{(2-2)} + \sum_{(2-3)} + \sum_{(2-4)}$$

where

$$
\begin{array}{lll}
(2-1): & N \ge 2^{k_1+2}, 2^{k_2+2} & \text{and} \quad k_3,k_4 \in \mathbb{N}\cup\{0\} \\
(2-2): & 2^{k_1+1} \ge N \ge 2^{k_2+2}, 2^{k_3+2}, 2^{k_4+2} & \text{and} \quad k_1 \ge k_2+3 \\
(2-3): & 2^{k_2+1} \ge N \ge 2^{k_1+2}, 2^{k_3+2}, 2^{k_4+2} & \text{and} \quad k_2 \ge k_1+3 \\
(2-4): & \text{otherwise} &
\end{array}
$$

Note that, by η_{k_j}, each variable ξ_j $(j=1,2,3,4)$ is restricted to the annulus $\{2^{k_j-1} \le |\xi_j| \le 2^{k_j+1}\}$.

In the case (2-1), since $|\xi_1| \le 2^{k_1} \le N/2$ and $|\xi_2| \le 2^{k_2+1} \le N/2$, we have $|\xi| = |\xi_1+\xi_2| \le N$. Hence $M_1 = 0$ and the integral vanishes.

If $2^{k_j} \gtrsim N$, then, from the relation

$$1 \sim \frac{|\xi_j|}{2^{k_j}} \lesssim \frac{|\xi_j|}{N},$$

we can derive the factor $1/N$ exchanging the differential $(-\Delta)^{\frac{1}{2}}$. In the case (2-4), at least two frequencies are greter than or similar to N and thus this case is harmless. So, we omit the estimate of (2-4). In the other cases, only one frequency is so. In particular, the case (2-3) contains the most complicated situation. So, for simplicity, we only consider the case (2-3).

Since $0 \leq m_2(\xi_2) \leq 1$, by trivial inequality,

$$M_1 = \left| \frac{m_1(\xi)}{m_2(\xi_2)} - 1 \right| \leq \frac{1}{m_2(\xi_2)} \leq C \left(\frac{2^{k_2}}{N} \right)^{1-s_2}$$

Moreover, clearly we have $M_2 = 1$.

Hence, using the relation $1 \sim |\xi_2|/2^{k_2} \lesssim |\xi_2|/N$, the considering integral is bounded by

$$\begin{aligned} &\frac{C}{N} \sum_{(2-3)} \int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \prod_{j=1}^{4} \eta_{k_j} \left\{ |\mathcal{F}_{x,t}[\psi_1](\xi_1, \tau_1)| \, |\xi_2| \, |\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}](\xi_2, \tau_2)| \right. \\ &\quad \left. \times |\mathcal{F}_{x,t}[\psi_1](\xi_3, \tau_3)| \, |\mathcal{F}_{x,t}[\chi_L \phi_2 + \overline{\chi_L \phi_2}](\xi_4, \tau_4)| \right\}. \end{aligned} \tag{31}$$

As stated above, we can not derive the expected factor $1/N^{\frac{3}{2}-\varepsilon}$ directoly. Our idea to overcome this difficulty is to compare the low frequency size to $N^{\frac{1}{2}}$, i.e. we split the case (2-3) into two cases:

$$\sum_{(2-3)} = \sum_{(2-3-i)} + \sum_{(2-3-ii)}$$

where

$$\begin{aligned} (2-3-i): &\quad (2-3) \quad \text{and} \quad 2^{\max\{k_1,k_3,k_4\}+3} \geq N^{\frac{1}{2}} \\ (2-3-ii): &\quad (2-3) \quad \text{and} \quad N^{\frac{1}{2}} \geq 2^{\max\{k_1,k_3,k_4\}+4}. \end{aligned}$$

In the case (2-3-i), we have $|\xi_j| \sim 2^{k_j} \gtrsim N^{\frac{1}{2}}$ for some $j \in \{1, 3, 4\}$. Hence we derive the additional factor $1/N^{\frac{1}{2}-\varepsilon}$, where $-\varepsilon$ is necessary for removing the characteristic function χ_L (cf. Lemma 4.2 below). Thus this case is harmless.

We consider the case (2-3-ii). In this case, we have

$$6 \max \left\{ \langle \tau_1 + |\xi_1|^2 \rangle, \langle \tau_2 \pm \langle \xi_2 \rangle \rangle, \langle \tau_3 + |\xi_3|^2 \rangle, \langle \tau_4 \pm \langle \xi_4 \rangle \rangle \right\} \geq |\xi_2|. \tag{32}$$

Indeed, since $2^{k_2+1} \geq N \geq 2^{2\max\{k_1,k_3,k_4\}+8}$, we have $|\xi_1|^2 + |\xi_3|^2 + |\xi_4| + 1 \leq 4 \cdot 2^{2(\max\{k_1,k_3,k_4\}+1)} \leq 2^{2\max\{k_1,k_3,k_4\}+4} \leq \frac{1}{16}N \leq \frac{1}{4}2^{k_2-1} \leq \frac{1}{4}|\xi_2|$ and thus

$$\begin{aligned}
& 4\max\left\{\langle\tau_1 + |\xi_1|^2\rangle, \langle\tau_2 \pm \langle\xi_2\rangle\rangle, \langle\tau_3 + |\xi_3|^2\rangle, \langle\tau_4 \pm \langle\xi_4\rangle\rangle\right\} \\
\geq\ & |\tau_1 + |\xi_1|^2| + |\tau - \tau_1 \pm \langle\xi_2\rangle| + |\tau_3 + |\xi_3|^2| + |\tau - \tau_3 \pm \langle\xi_4\rangle| \\
\geq\ & |\tau_1 + |\xi_1|^2 + (\tau - \tau_1 \pm \langle\xi_2\rangle) - (\tau_3 + |\xi_3|^2) - (\tau - \tau_3 \pm \langle\xi_4\rangle)| \\
=\ & \left||\xi_1|^2 \pm \langle\xi_2\rangle - |\xi_3|^2 \pm \langle\xi_4\rangle\right| \\
\geq\ & |\xi_2| - (|\xi_1|^2 + |\xi_3|^2 + |\xi_4| + 1) \\
\geq\ & |\xi_2| - \frac{1}{4}|\xi_2| = \frac{3}{4}|\xi_2|.
\end{aligned}$$

Hence (32) follows.

Then the considering integral, which is subcase of (31), is bounded by

$$\frac{C}{N^{\frac{3}{2}-\varepsilon}} \sum_{(2-3-ii)} \int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \prod_{j=1}^{4} \eta_{k_j} \cdots \leq \frac{C}{N^{\frac{3}{2}-\varepsilon}} \int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \cdots, \tag{33}$$

where $\cdots$ denotes the integrand

$$\begin{aligned}
& \max\left\{\langle\tau_1 + |\xi_1|^2\rangle, \langle\tau_2 \pm \langle\xi_2\rangle\rangle, \langle\tau_3 + |\xi_3|^2\rangle, \langle\tau_4 \pm \langle\xi_4\rangle\rangle\right\}^{\frac{1}{2}(1-\varepsilon)} \\
& \times \left\{|\mathcal{F}_{x,t}[\psi_1](\xi_1,\tau_1)|\,|\xi_2|\,\left|\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}](\xi_2,\tau_2)\right|\,|\mathcal{F}_{x,t}[\psi_1](\xi_3,\tau_3)|\,\left|\mathcal{F}_{x,t}[\chi_L\phi_2 + \overline{\chi_L\phi_2}](\xi_4,\tau_4)\right|\right\}.
\end{aligned}$$

Then, deviding the integral according to the maximal Bourgain weight and using the Lemma 4.1 below, we obtain the bound

$$\frac{C}{N^{\frac{3}{2}-\varepsilon}} \|\psi_1\|_{X^{1,\alpha}}^2 \|\phi_2\|_{Y^{1,\beta}}^2.$$

This implies the expected bound (28) and hence Proposition 3.2 follows.

Lemma 4.1
Let $\alpha, \beta > 1/2$ and $\varepsilon > 0$. We consider the following integrals.

$$\int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} \langle\tau_1 + |\xi_1|^2\rangle^{\frac{1}{2}(1-\varepsilon)} |\mathcal{F}_{x,t}[\psi_1]|\,|\xi_2|\,\left|\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}]\right|\,|\mathcal{F}_{x,t}[\psi_1]|\,\left|\mathcal{F}_{x,t}[\chi_L\phi_2 + \overline{\chi_L\phi_2}]\right|,$$

$$\int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} |\mathcal{F}_{x,t}[\psi_1]|\,|\xi_2|\langle\tau_2 \pm \langle\xi_2\rangle\rangle^{\frac{1}{2}(1-\varepsilon)} \left|\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}]\right|\,|\mathcal{F}_{x,t}[\psi_1]|\,\left|\mathcal{F}_{x,t}[\chi_L\phi_2 + \overline{\chi_L\phi_2}]\right|,$$

$$\int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} |\mathcal{F}_{x,t}[\psi_1]|\,|\xi_2|\,\left|\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}]\right|\,\langle\tau_3 + |\xi_3|^2\rangle^{\frac{1}{2}(1-\varepsilon)} |\mathcal{F}_{x,t}[\psi_1]|\,\left|\mathcal{F}_{x,t}[\chi_L\phi_2 + \overline{\chi_L\phi_2}]\right|,$$

$$\int_{\substack{\xi=\xi_{12}=\xi_{34} \\ \tau=\tau_{12}=\tau_{34}}} |\mathcal{F}_{x,t}[\psi_1]|\,|\xi_2|\,\left|\mathcal{F}_{x,t}[\phi_2 + \overline{\phi_2}]\right|\,|\mathcal{F}_{x,t}[\psi_1]|\,\langle\tau_4 \pm \langle\xi_4\rangle\rangle^{\frac{1}{2}(1-\varepsilon)} \left|\mathcal{F}_{x,t}[\chi_L\phi_2 + \overline{\chi_L\phi_2}]\right|.$$

Then all of them are bounded by

$$C\|\psi_1\|_{X^{1,\alpha}}^2 \|\phi_2\|_{Y^{1,\beta}}^2$$

where C is independent of $L := [t_0, t_1]$ and N.

Lemma 4.1 is a direct consequence of Sobolev's embedding theorem, Strichartz type estimate (see [8], Lemma 2.4) and the characteristic function lemma below.

Lemma 4.2 (characteristic function lemma)
Let $s \in \mathbb{R}$, $\varepsilon > 0$, $\alpha > 1/2$ and L be an interval in $\mathbb{R}$ with the length $|L| \leq 1$. Further let χ_L be the characteristic function on L. Then we have

$$\|\chi_L u\|_{X^{s,\frac{1}{2}-\varepsilon}} \leq C\|u\|_{X^{s,\alpha}}, \tag{34}$$

$$\|\chi_L v\|_{Y^{s,\frac{1}{2}-\varepsilon}} \leq C\|v\|_{Y^{s,\alpha}}. \tag{35}$$

where C depends only on ε and α. We may replace X and Y with X_- and Y_-, respectively.

Proof of Lemma 4.2.
We have, for any $\alpha > 1/2$,

$$\|\chi_L h\|_{H_t^{\frac{1}{2}-\varepsilon}} \leq C\|h\|_{H_t^\alpha} \tag{36}$$

for some constant $C > 0$ depending only on ε and α. This inequality is analoge to [11] Lemma 3.2. From (36), we have

$$\begin{aligned}
\|\chi_L u\|_{X^{s,\frac{1}{2}-\varepsilon}} &= \left\|(1-\Delta)^{\frac{s}{2}}(1-\partial_t)^{\frac{1}{2}(\frac{1}{2}-\varepsilon)}U(-\cdot)(\chi_L u)\right\|_{L_t^2 L_x^2} \\
&= \left\|\left\|\chi_L\left[(1-\Delta)^{\frac{s}{2}}U(-\cdot)u\right]\right\|_{H_t^{\frac{1}{2}-\varepsilon}}\right\|_{L_x^2} \\
&\leq C\left\|\left\|(1-\Delta)^{\frac{s}{2}}U(-\cdot)u\right\|_{H_t^\alpha}\right\|_{L_x^2} \\
&= C\|u\|_{X^{s,\alpha}}.
\end{aligned}$$

Similarly, from (36), we have $\|\chi_L v\|_{Y^{s,\frac{1}{2}-\varepsilon}} \leq \|v\|_{Y^{s,\alpha}}$.
Hence we have done. □

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We assume the conditions of Proposition 3.2 For simplicity, we only give the proof for the dimension $d = 3, 4$. In the case $d = 1, 2$, we need some minor modifications.

Now we give the proof for $d = 3, 4$. Set

$$A_{1,2}(t) := \|I_1\psi(t)\|_{\dot{H}^1(\mathbb{R}^d)} + \|I_2\phi(t)\|_{H^1}.$$

Then, by Proposition 3.1 (18) (we also use (17) as $\|I_2\phi(t)\|_{L^2} \leq \|\phi(t)\|_{L^2}$.),

$$\begin{aligned}
A_{1,2}(t) &\leq 2N^{1-s_1}\|\psi(t)\|_{\dot{H}^{s_1}(\mathbb{R}^d)} + 4N^{1-s_2}\|\phi(t)\|_{H^{s_2}(\mathbb{R}^d)} \\
&\leq 4N^{1-\min\{s_1,s_2\}}\left(\|\psi(t)\|_{\dot{H}^{s_1}(\mathbb{R}^d)} + \|\phi(t)\|_{H^{s_2}(\mathbb{R}^d)}\right).
\end{aligned} \tag{37}$$

In particular, we have

$$A_{1,2}(0) \leq 4C_0 N^{1-\min\{s_1,s_2\}}, \tag{38}$$

where

$$C_0 := \|\psi_0\|_{\dot{H}^{s_1}(\mathbf{R}^d)} + \|\phi_0\|_{H^{s_2}(\mathbf{R}^d)}.$$

Now let (ψ,ϕ) be a solution of (KGS) on $L := [t_0, t_0+\delta]$. Then, by Lemma 1.2, we find that, for any $\theta_{1,2} < \dfrac{4+2s_2-d}{4}$, there exist $\alpha,\beta > 1/2$ such that

$$\|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)} \leq C_\rho A_{1,2}(t_0) + C'_\rho \delta^{\theta_{1,2}} \left(\|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)}\right)^2 \tag{39}$$

for some $C_\rho, C'_\rho \geq 1$ both independent of L and N, where ρ is an arbitrary fixed time smooth cut-off function introduced in the section 1.

Then, consider the quadratic equation $x \leq C_\rho A_{1,2}(t_0) + C'_\rho \delta^{\theta_{1,2}} x^2$. By the continuity of $x = x(\delta) := \|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)}$ in δ (t_0 is fixed), we have, for any $\nu > 1$,

$$\|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)} \leq 2\nu C_\rho A_{1,2}(t_0), \tag{40}$$

if we take

$$\delta \leq \left(4\nu C_\rho C'_\rho A_{1,2}(t_0)\right)^{-\frac{1}{\theta_{1,2}}}. \tag{41}$$

Moreover, by Gagliardo-Nirenberg inequality (and using the condition (4) if $d=4$), we find that

$$E_{1,2}(\psi,\phi)(t) \leq C'_0 (A_{1,2}(t))^2 \tag{42}$$

for some $C'_0 \geq 1$ depending only on $\|\psi_0\|_{L^2(\mathbf{R}^d)}$. Also, we find that

$$A_{1,2}(t) \leq \widetilde{C}\sqrt{E_{1,2}(\psi,\phi)(t)} \tag{43}$$

for some $\widetilde{C} \geq 1$ depending only on $\|\psi_0\|_{L^2(\mathbf{R}^d)}$.

We show that the local solution of (KGS) can be continued until any given $T > 0$, which completes the proof of Theorem 1.1.

For this, let us make the following observation. We first assume the following:
Assumption : For any given $T > 1$, there exists a solution (ψ,ϕ) on $[0,T]$ such that

$$A_{1,2}(t) \leq \Omega A_{1,2}(0), \quad \forall t \in [0,T]$$

for some constant $\Omega > 0$ determined later.

Now, for fixed $\nu > 1$, we set

$$\delta_0 := \left(4\nu C_\rho C'_\rho \Omega A_{1,2}(0)\right)^{-\frac{1}{\theta_{1,2}}}. \tag{44}$$

We may assume that $\kappa := T/\delta_0 \in \mathbb{N}$ by the suitable choice of ν. Then we set $L_j := [(1-j)\delta, j\delta]$ $(j=1,2,\cdots,\kappa)$ and thus $[0,T] = L_1 \cup L_2 \cup \cdots \cup L_\kappa$. Moreover, we may assume that $\delta_0 \le 1$. Indeed, by Proposition 3.1 (19), $\|\phi_0\|_{H^{s_2}} \le \|I_2\phi_0\|_{H^1} \le A_{1,2}(0)$. Thus, if $\|\phi_0\|_{H^{s_2}} \le 1$, then take $\nu \ge 1/\|\phi_0\|_{H^{s_2}}$ and otherwise, automatically $\delta_0 \le 1$.

On each interval L_j, we have (39) replacing L with L_j and $A_{1,2}(t_0)$ with $A_{1,2}((j-1)\delta)$ which is bounded by $\Omega A_{1,2}(0)$. Thus, from (41, 40), it follows that

$$\|I_1\psi\|_{X^{1,\alpha}(L_j)} + \|I_2\phi\|_{Y^{1,\beta}(L_j)} \le 2\nu C_\rho \Omega A_{1,2}(0) \quad (\forall j = 1,2,\cdots,\kappa). \tag{45}$$

Then, by Proposition 3.2

$$\begin{aligned} E_{1,2}(\psi,\phi)(T) &= E_{1,2}(\psi,\phi)(\kappa\delta) \\ &= E_{1,2}(\psi,\phi)(\kappa\delta) - E_{1,2}(\psi,\phi)((\kappa-1)\delta) \\ &\quad + E_{1,2}(\psi,\phi)((\kappa-1)\delta) - E_{1,2}(\psi,\phi)((\kappa-2)\delta) \\ &\quad + \cdots\cdots\cdots \\ &\quad + E_{1,2}(\psi,\phi)(\delta) - E_{1,2}(\psi,\phi)(0) + E_{1,2}(\psi,\phi)(0) \\ &= C_*\left\{\frac{1}{N^{1-\varepsilon}}Q(L_\kappa)^3 + \frac{1}{N^{\frac{3}{2}-\varepsilon}}Q(L_\kappa)^4\right\} \\ &\quad + C_*\left\{\frac{1}{N^{1-\varepsilon}}Q(L_{\kappa-1})^3 + \frac{1}{N^{\frac{3}{2}-\varepsilon}}Q(L_{\kappa-1})^4\right\} \\ &\quad + \cdots\cdots\cdots \\ &\quad + C_*\left\{\frac{1}{N^{1-\varepsilon}}Q(L_1)^3 + \frac{1}{N^{\frac{3}{2}-\varepsilon}}Q(L_1)^4\right\} + E_{1,2}(\psi,\phi)(0) \end{aligned} \tag{46}$$

where $Q(L) := \|I_1\psi\|_{X^{1,\alpha}(L)} + \|I_2\phi\|_{Y^{1,\beta}(L)}$.

By (45), (38) and (42),

$$\begin{aligned} &\text{R.H.S. of (46)} \\ &\le \kappa C_*\left\{\frac{1}{N^{1-\varepsilon}}(2\nu C_\rho\Omega A_{1,2}(0))^3 + \frac{1}{N^{\frac{3}{2}-\varepsilon}}(2\nu C_\rho\Omega A_{1,2}(0))^4\right\} + C_0'(A_{1,2}(0))^2 \\ &\le \kappa C_*(A_{1,2}(0))^2\left\{\frac{1}{N^{1-\varepsilon}}(2\nu C_\rho\Omega)^3\left(4C_0N^{1-\min\{s_1,s_2\}}\right)\right. \\ &\qquad \left. + \frac{1}{N^{\frac{3}{2}-\varepsilon}}(2\nu C_\rho\Omega)^4\left(4C_0N^{1-\min\{s_1,s_2\}}\right)^2\right\} + C_0'(A_{1,2}(0))^2. \end{aligned} \tag{47}$$

Since, by (44) and (38),

$$\kappa = \frac{T}{\delta_0} = T(4\nu C_\rho C_\rho'\Omega A_{1,2}(0))^{\frac{1}{\theta_{1,2}}} \le T(16\nu C_\rho C_\rho' C_0\Omega)^{\frac{1}{\theta_{1,2}}}N^{\frac{1-\min\{s_1,s_2\}}{\theta_{1,2}}},$$

$$\begin{aligned} &\text{R.H.S. of (47)} \\ &\le TC_*(16\nu C_\rho C_\rho' C_0\Omega)^{\frac{1}{\theta_{1,2}}}(A_{1,2}(0))^2\left\{32C_0(\nu C_\rho\Omega)^3N^{(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)}\right. \\ &\quad \left. + 256C_0^2(\nu C_\rho\Omega)^4N^{(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-(\frac{3}{2}-\varepsilon)}\right\} + C_0'(A_{1,2}(0))^2. \end{aligned} \tag{48}$$

Here, by (43), we have $(A_{1,2}(T))^2 \leq \widetilde{C}^2 E_{1,2}(\psi,\phi)(T)$ and thus, in order that $A_{1,2}(T) \leq \Omega A_{1,2}(0)$, we need that

$$E_{1,2}(\psi,\phi)(T) \leq R.H.S.of(48) \leq \frac{\Omega^2}{\widetilde{C}^2}(A_{1,2}(0))^2. \tag{49}$$

For this, choose Ω such that

$$\Omega \geq \widetilde{C}\sqrt{2C_0'}. \tag{50}$$

Then it is required that

$$\begin{aligned}\frac{\Omega^2}{2\widetilde{C}^2} \quad &\geq \quad TC_*(16\nu C_\rho C_\rho' C_0 \Omega)^{\frac{1}{\theta_{1,2}}}\left\{32C_0(\nu C_\rho \Omega)^3 N^{(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)}\right. \\ &\left. +256C_0^2(\nu C_\rho\Omega)^4 N^{(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-\left(\frac{3}{2}-\varepsilon\right)}\right\}. \end{aligned} \tag{51}$$

To realize (51), all powers of N must be negative, i.e.

$$(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)<0 \tag{52}$$

and

$$(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-\left(\frac{3}{2}-\varepsilon\right)<0. \tag{53}$$

Then, taking N sufficiently large, we realize (51). Note that

$$\begin{aligned}&(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)-\left[(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-\left(\frac{3}{2}-\varepsilon\right)\right] \\ &=\min\{s_1,s_2\}-\frac{1}{2}.\end{aligned}$$

Moreover recall that we are assuming that $1 \geq s_1 > 1/2$ and $1 \geq s_2 > 0$. Thus if $1/2 \geq s_2$, then we need (53), i.e.

$$(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-\left(\frac{3}{2}-\varepsilon\right)=(1-s_2)\left(2+\frac{1}{\theta_{1,2}}\right)-\left(\frac{3}{2}-\varepsilon\right)<0.$$

Since we can take $\theta_{1,2}$ and ε arbitrarlily close to $\frac{2s_2+4-d}{4}$ and 0, respectively (taking both α and β close to 1/2), we find that we need at least that $s_2 > 1/2$, which is impossible.

On the other hand, if $s_2 > 1/2$, then (52) is required. For this, we need that

$$\min\{s_1,s_2\} > \frac{4}{8+2s_2-d},$$

which is the condition of Theorem 1.1.

Hence, we obtain (49) if we take s_1, s_2 as in Theorem 1.1 and N so large that

$$N^{(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)}\left[TC_*(16\nu C_\rho C'_\rho C_0\Omega)^{\frac{1}{\theta_{1,2}}}32C_0(\nu C_\rho\Omega)^3\right] \le \frac{\Omega^2}{4\widetilde{C}^2} \tag{54}$$

and

$$N^{(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-(\frac{3}{2}-\varepsilon)}\left[TC_*(16\nu C_\rho C'_\rho C_0\Omega)^{\frac{1}{\theta_{1,2}}}256C_0^2(\nu C_\rho\Omega)^4\right] \le \frac{\Omega^2}{4\widetilde{C}^2}. \tag{55}$$

From the above observation, we determine the parameters Ω, s_1, s_2 and N as in (50), Theorem 1.1 and (54, 55), respectively. Then we show that the solution (ψ, ϕ) exists on $[0, T]$ for any given $T > 0$ and satisfies that

$$\begin{aligned} &\|\psi(t)\|_{H^{s_1}(\mathrm{R}^d)} + \|\phi(t)\|_{H^{s_2}(\mathrm{R}^d)} \\ \le\ & \|\psi_0\|_{L^2(\mathrm{R}^d)} + \Omega A_{1,2}(0)\ \left(\le \|\psi_0\|_{L^2(\mathrm{R}^d)} + 4C_0N^{1-\min\{s_1,s_2\}}\right), \end{aligned} \tag{56}$$

which completes the proof.

Note first that, by the locally well-posed result, there exists $\delta_1 > 0$ such that the solution exists on $[0, \delta_1]$. On the other hand, if we have the bound (56) at the initial time, we can extend the existence interval by some length δ_2.

Now we set

$$\delta^* := \min\{\delta_0, \delta_1, \delta_2\}. \tag{57}$$

Then, taking ν sufficiently large, we can take $\delta_* = \delta_0$ and therefore $\kappa = T/\delta_* \in \mathbb{N}$. By (41, 40), we have

$$Q(L_1) \le 2\nu C_\rho A_{1,2}(0) \le 2\nu C_\rho \Omega A_{1,2}(0). \tag{58}$$

Then, by the same argument as above,

$$\begin{aligned} &E_{1,2}(\psi,\phi)(\delta^*) \\ &= E_{1,2}(\psi,\phi)(\delta^*) - E_{1,2}(\psi,\phi)(0) + E_{1,2}(\psi,\phi)(0) \\ &\le C_*(A_{1,2}(0))^2\left\{\frac{1}{N^{1-\varepsilon}}(2\nu C_\rho\Omega)^3\left(4C_0N^{1-\min\{s_1,s_2\}}\right)\right. \\ &\qquad \left. + \frac{1}{N^{\frac{3}{2}-\varepsilon}}(2\nu C_\rho\Omega)^4\left(4C_0N^{1-\min\{s_1,s_2\}}\right)^2\right\} + C'_0(A_{1,2}(0))^2 \\ &\le \kappa C_*(A_{1,2}(0))^2\left\{\frac{1}{N^{1-\varepsilon}}(2\nu C_\rho\Omega)^3\left(4C_0N^{1-\min\{s_1,s_2\}}\right)\right. \\ &\qquad \left. + \frac{1}{N^{\frac{3}{2}-\varepsilon}}(2\nu C_\rho\Omega)^4\left(4C_0N^{1-\min\{s_1,s_2\}}\right)^2\right\} + C'_0(A_{1,2}(0))^2 \\ &\le TC_*(16\nu C_\rho C'_\rho C_0\Omega)^{\frac{1}{\theta_{1,2}}}(A_{1,2}(0))^2\left\{32C_0(\nu C_\rho\Omega)^3N^{(1-\min\{s_1,s_2\})\left(1+\frac{1}{\theta_{1,2}}\right)-(1-\varepsilon)}\right. \\ &\quad \left. +256C_0^2(\nu C_\rho\Omega)^4N^{(1-\min\{s_1,s_2\})\left(2+\frac{1}{\theta_{1,2}}\right)-(\frac{3}{2}-\varepsilon)}\right\} + C'_0(A_{1,2}(0))^2. \end{aligned} \tag{59}$$

From the choice of parameters Ω, s_1, s_2 and N (cf. (50), (54 , 55)), we have

$$A_{1,2}(\delta^*) \leq \Omega A_{1,2}(0) \tag{60}$$

and thus, by Proposition 3.1 and the L^2-conservation law, we have the bound (56) for the time $t = \delta^*$. Hence we extend the existence interval to $[0, 2\delta^*]$.

Next we consider $E_{1,2}(\psi, \phi)(2\delta^*)$. By (60) and the same way as above, we have

$$A_{1,2}(2\delta^*) \leq \Omega A_{1,2}(0)$$

and we can extend the existence interval to $[0, 3\delta^*]$.

We can continue this procedure until the time T and thus we have shown that the solution exists on $[0, T]$ for any given $T > 0$. We have done. $\square$

6 Further result

In this section, we consider the wave-Schrödinger system below, which is the massles version of the Klein-Gordon-Schrödinger system.

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u & = & 2vu, \\ \partial_t^2 v - \Delta v & = & -|u|^2, \end{array} \right.$$

where u and v are complex and real valued functions on $\mathbb{R}^d \times [0, \infty)$, respectively.

As the Klein-Gordon-Schrödinger system, this system is transformed into a time first oreder system

$$\text{(WS)} \left\{ \begin{array}{rcll} i\partial_t\psi + \Delta\psi & = & (\phi + \overline{\phi})\psi, & x \in \mathbb{R}^d, \quad t \geq 0, \\ i\partial_t\phi - (-\Delta)^{\frac{1}{2}}\phi & = & (-\Delta)^{-\frac{1}{2}}(|\psi|^2), & x \in \mathbb{R}^d, \quad t \geq 0, \\ \psi(0) & = & \psi_0, & x \in \mathbb{R}^d, \\ \phi(0) & = & \phi_0, & x \in \mathbb{R}^d, \end{array} \right.$$

where both ψ and ϕ are complex valued functions.

The main difference from the massive case (KGS) is the treatment of the low frequency part. Indeed, we no longer have the L^2-bound for the wave equation and therefore we have to work with the homogeneous Sobolev spaces $\dot{H}^s$ ($s \leq 1$) in order to show the global well-posedness. At that time, since it is not true that $\|g\|_{\dot{H}^s} \lesssim \|I_N^s g\|_{\dot{H}^1}$, the bound for the modified energy does not imply the one for the $\dot{H}^s$-norm of the solution. To overcome this difficulty, we introduce the space $\Omega^{s,b}$. We set

$$\omega^{s,b}(\xi) := \left\{ \begin{array}{lll} |\xi|^b & \text{if} & |\xi| \leq 1, \\ |\xi|^s & \text{if} & 1 \leq |\xi|, \end{array} \right. \tag{61}$$

and define the operator $D^{s,b}$ by

$$\mathcal{F}_x[D^{s,b}f](\xi) := w^{s,b}(\xi)\mathcal{F}_x[f](\xi). \tag{62}$$

Let $\mathcal{Z}(\mathbb{R}^d) := \left\{f \in \mathcal{S}(\mathbb{R}^d) \,\middle|\, (D^\alpha \mathcal{F}_x[f])(0) = 0, \forall \alpha \in (\mathbb{N} \cup \{0\})^d \right\}$. We find that $\mathcal{Z}' = \mathcal{S}'/\mathcal{P}$ where $\mathcal{P}$ is the space of all polynomials. We define the space $\Omega^{s,b}(\mathbb{R}^d)$ by

$$\Omega^{s,b}(\mathbb{R}^d) := \left\{ f \in \mathcal{Z}'(\mathbb{R}^d) \,\middle|\, D^{s,b} f \in L^2(\mathbb{R}^d) \right\}. \tag{63}$$

Then we find that $\|f\|_{\Omega^{s,1}} := \|D^{s,1} f\|_{L^2} \sim \|I_N^s f\|_{\dot{H}^1}$ and thus we can prove the global well-posedness below the energy class. Moreover, introducing the modified multiplier for I-method, we can prove more general result.

Let $m_{N,M}^{s,b} \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$ be radial, non-increasing and

$$m_{N,M}^{s,b}(\xi) = \begin{cases} (M|\xi|)^{b-1} & \text{if} \quad |\xi| \le 1/M, \\ 1 & \text{if} \quad 1/M \le |\xi| \le N, \\ smooth & \text{if} \quad N \le |\xi| \le 2N, \\ (N/|\xi|)^{1-s} & \text{if} \quad 2N \le |\xi|. \end{cases}$$

We define the operator $I_{N,M}^{s,b}$ by

$$\mathcal{F}_x[I_{N,M}^{s,b} f](\xi) := m_{N,M}^{s,b}(\xi) \mathcal{F}_x[f](\xi). \tag{64}$$

In particular, we define $I_{N,M}^{1,1} f := f$.

Note that we have $\|I_{N,M}^{s,b} f\|_{\dot{H}^1} \sim \|f\|_{\Omega^{s,b}} := \|D^{s,b} f\|_{L^2}$.

Then, with some low frequency analysis, we obtain the following result.

Theorem 6.1 (Global well-posedness)
Let $d = 3, 4$. Assume (4) when $d = 4$. If s_1 and s_2 satisfy that

$$1 \ge s_1, s_2 > \frac{4}{8 + 2s_2 - d}, \tag{65}$$

and b satisfies that

$$b \le \frac{1}{2}(3 - p_d) \ (d = 3), \quad b \le \frac{1}{3}(5 - 2p_d) \ (d = 4), \tag{66}$$

where

$$q_d := \frac{\sqrt{(8-d)^2 + 32} - (8-d)}{4}, \quad p_d := \begin{cases} q_d & \text{if} \quad s_1 \ge \frac{4}{8+q_d-d} \\ \frac{4}{s_1} + d - 8 & \text{if} \quad \frac{4}{8+q_d-d} > s_1 > \frac{4}{9-d} \end{cases},$$

then (WS) is globally well-posed for the data $(\psi_0, \phi_0) \in H^{s_1}(\mathbb{R}^d) \times \Omega^{s_2,b}(\mathbb{R}^d)$.

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